Proving a language not regular

- direct argument
- Pumping Lemma
- closure properties

Pumping Lemma

Let L be a regular lg.

There is an integer p such That

for every string $s \in L$

with $|s| \geq p$

There are strings x, y, z

s.t.

- $s = xyz$
- $|xy| \leq p$
- $|y| \geq 1$
- $\forall i \geq 0, \; xy^iz \in L$

Proof:

Let L be regular.

Let D be a DFA recognizing L.

Let p be # of states in D

Pick a string S s.t. $S \in L$ and $|S| \geq p$.

$S = S_1 \ldots\ldots S_n$ $\quad$ S_i individual chars

Let $r_1, \ldots\ldots, r_{n+1}$ be the sequence of states on accepting S

n+1 states in sequence

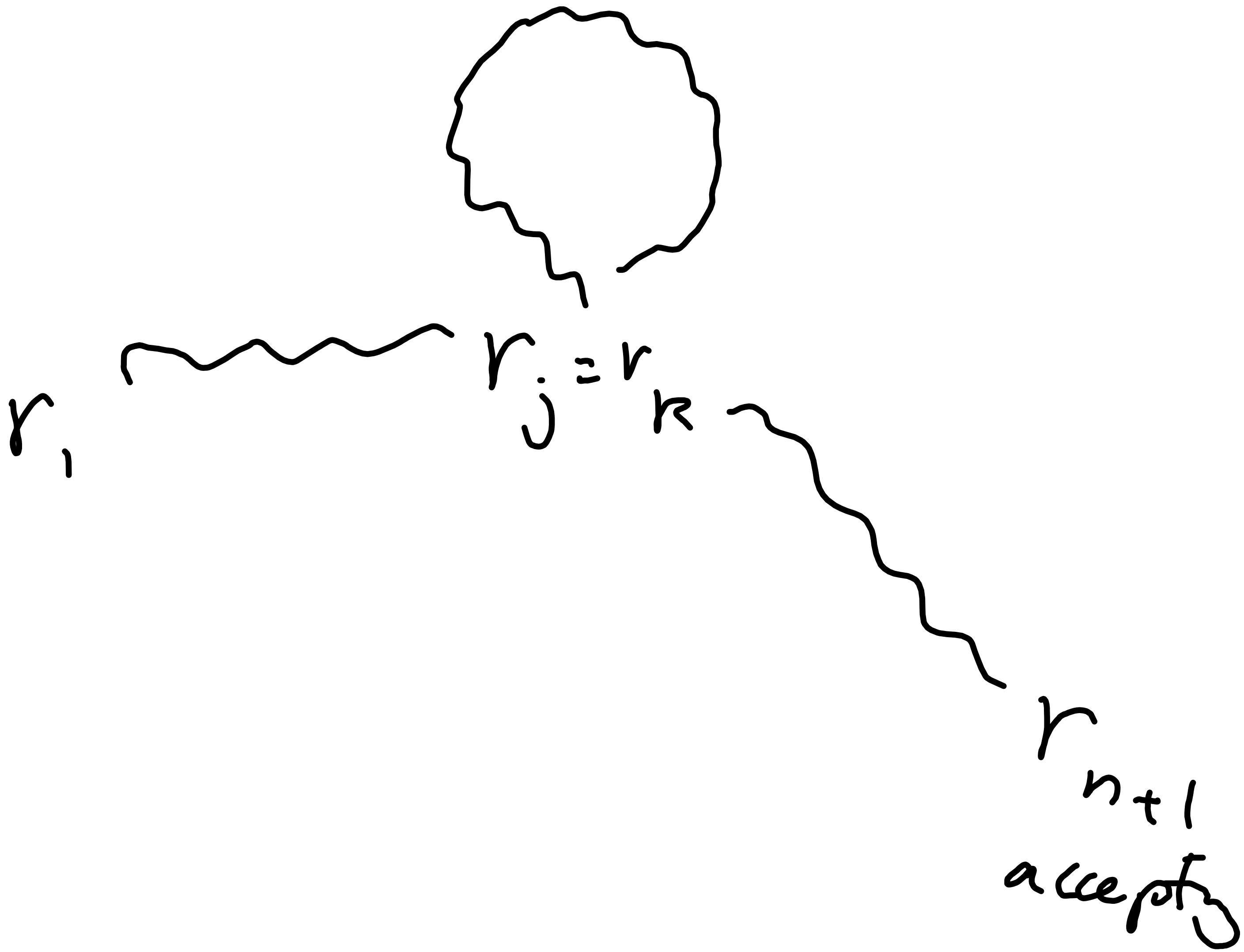
$n = |s| \geq p$ = # of states in DFA D

so 2 states in sequence are the same. Let's make them be

r_j and r_k $j \neq k$

can pick $j < k$

can pick in first p+1 states of state sequence


r_1
$r_j = r_k$
r_{n+1}
accepting

$$s_1 \; s_2 \; \ldots\ldots \; s_{j-1} \; s_j \; \ldots\ldots \; s_{k-1} \; s_k \; \ldots\ldots \; s_n$$

$$r_1 \quad r_2 \qquad\qquad r_j \qquad\qquad r_k \qquad\qquad r_{n+1}$$

So let $x = s_1 \ldots\ldots s_{j-1}$

$y = s_j \ldots\ldots s_{k-1}$

$z = s_k \ldots\ldots s_n$

$$j < k$$

$$r_j = r_k$$

$L = \{0^n 1^n : n \geq 0\}$

is not regular

Proof by contradiction

Assume L is regular

Invoke PL

and show PL doesn't hold for L

$|xy| \leq p$ because

we can force r_j & r_k

to be in first $p+1$

states so $k \leq p+1$

so

$|xy| = k-1 \leq p$

Pumping Lemma

Let L be a regular lg.

~~There is an integer~~ for every p such that

~~for every~~ There is a string $s \in L$

with $|s| \geq p$

~~There are~~ for every strings x, y, z

s.t.

- $s = xyz$
- $|xy| \leq p$
- $|y| \geq 1$
- $\exists$ $\cancel{\forall}\, i \geq 0$, $xy^iz \notin L$

$L = \{ 0^n 1^n \}$ is not regular

Proof:

Let p be the integer given by the pumping Lemma

Consider $S_p = 0^p 1^p$

Clearly $S_p \in L$ and $|S_p| \geq p$

So by PL, there are strings x, y, z

s.t. $|xy| \leq p$

$|y| \geq 1$

$s = 0^p 1^p = xyz$

and $xy^iz \in L, \forall i$

which satisfy the PL conditions

Since $|xy| \le p$ and $s = 0^p 1^p$

y (and x) consists of all zeros

So $x\,y\,z$ looks like

$$\underbrace{0^r}_{x}\ \underbrace{0^q}_{y}\ \underbrace{0^{t-p}\,1^p}_{z}$$

$r = |x|$

$q = |y|$

$t = |z|$

$r + q + (t - p) = p$

So $x y^i z^p$

$= 0^r\, \underline{0^{iq}}\, 0^{t-p}\, 1^p$

Since $q = |y| \ge 1$, so

$2q \ne q$

So increasing i adds zeros to the start of string. extra copy of q

So $xy^2z = 0^{r+2q+t-p}\, 1^p$

$r+2q+t-p \ne p$

So $xy^2z \notin L$

Claim: $\{ww : w \in \{0,1\}^*\}$ is not regular

Proof: Let p be from PL

Pick $s_p = 0^p 1 0^p 1$

separation

x, y will have to be all zeros

Let x, y, z satisfy the PL conditions

$s_p = xyz$

xy^iz

y is all zeros

y lies in 1st group of zeros

So adding extra copies of y changes the length of 1st group of zeros, But not the second

$L = \{ w \in \{0,1\}^* \text{ s.t. } w \text{ has equal \#s of 0's and 1's} \}$

Consider $L \cap 0^* 1^*$

$= 0^n 1^n$

not regular

Therefore L not regular

Proof by contradiction

Assume L is regular.
We know 0^*1^* is regular
Then $L \cap 0^*1^*$ is
regular by closure
under $\cap$

but $L \cap 0^*1^* = \{0^n 1^n\}$
which we know is not
regular

contradiction

Claim: $\{a^n b^n\} = L$

is not regular

Proof: Suppose L is regular

Consider $h(L)$ is regular

where h: $a \to 0$

$b \to 1$

but $h(L) = \{0^n 1^n\}$

which isn't regular

$$\{(ab)^n c^n\}$$

$$h: \quad a \to 0$$
$$b \to \varepsilon$$
$$c \to 1$$

$L = \{$balanced strings of parenteces$\}$

$((()())))$

- first $\cap$ with $\{(^*)^*\}$

$\Rightarrow \{(^n)^n\}$

- use homomorphism

h: $(\longrightarrow 0$

$) \longrightarrow 1$

$\Rightarrow \{0^n 1^n\}$