Please do The
HBS web
Survey. (See
news group.)

Equivalence relations.

Sets — A set S

$R \subseteq S \times S$ — binary relation.

Eg. $S = \{a, b, c\}$

$R = \{(a,b), (b,a)\}$

Eq. relation

$R \subseteq S \times S$:

1) $\forall x \in S$. $(x,x) \in R$
2) $\forall x, y \in S$. $(x,y) \in R \Rightarrow (y,x) \in R$
3) $\forall x, y, z \in S$. $(x,y) \in R$, $(y,z) \in R$, $\Rightarrow (x,z) \in R$

Eg. $(\mathbb{N}, =)$

$(\mathbb{N} \cup \{0\}, R)$

$R = \{(x, y) \mid x - y \text{ is divisible by } 5\}$.

Reflexive: $R(x, x)$: $x - x = 0$ ✓

$R(x, y) \Rightarrow R(y, x)$ $\quad 5 \mid x - y$

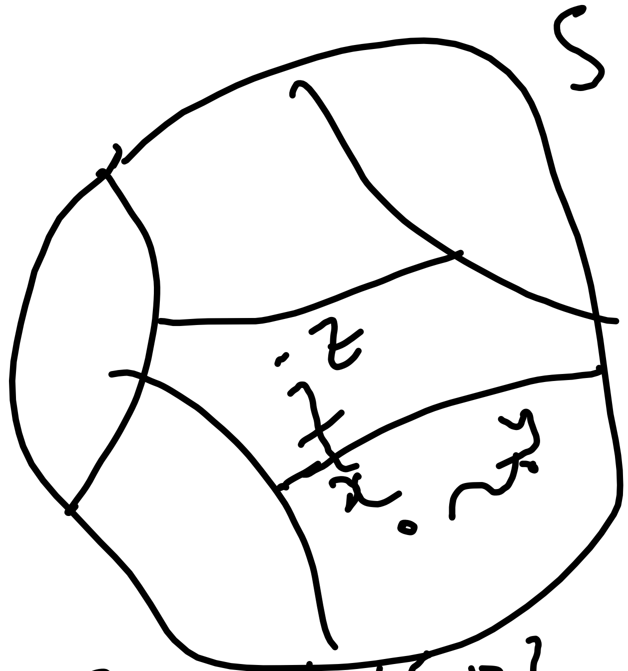
$5 \mid y - x$

$R(x, y) \wedge R(y, z)$ $\quad 5 \mid x - y \wedge 5 \mid y - z$

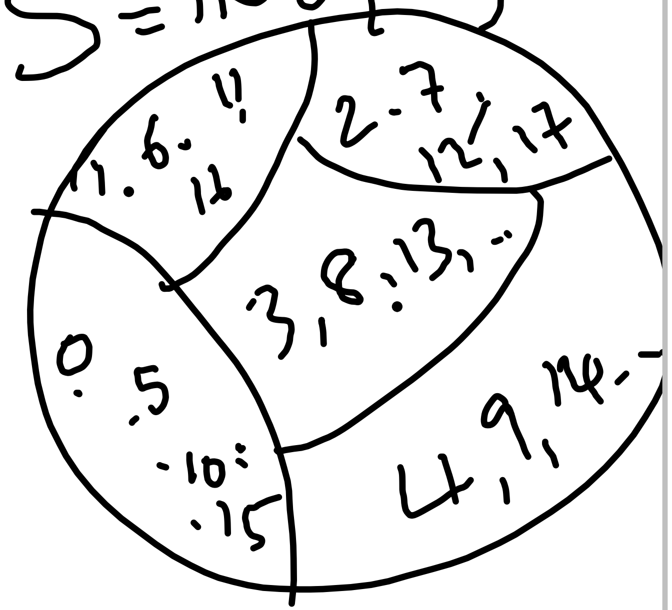
$5 \mid x - y + y - z \quad 5 \mid x - z \quad R(x, z)$

$\sim$ — Eq rel on S

$\sim_5$ $R(x,y)$ if $5 \mid x-y$



$S = \mathbb{N} \cup \{0\}$



$\sim$ - Eq rel on S

$$[x]_\sim = \{ y \in S \mid x \sim y \} \quad x \in S$$

$$[x]_\sim \subseteq S$$

$$E = \{ [x]_\sim \mid x \in S \}$$

Claim: E partitions S

$[0]_{\sim_5} = \{0, 5, 10, \ldots\}$

$E = \{ [0], [1], \ldots [4] \}$

E is a partition of S.

- $x \in S$. $x \in X \in E$

$x \in [x] \in E$.

- Sets in E are disjoint.

Let $[x] \cap [y] \neq \emptyset$

$\cancel{\alpha}$ $z \in [x] \cap [y]$

$z \sim x$ & $z \sim y$

$x \sim y$

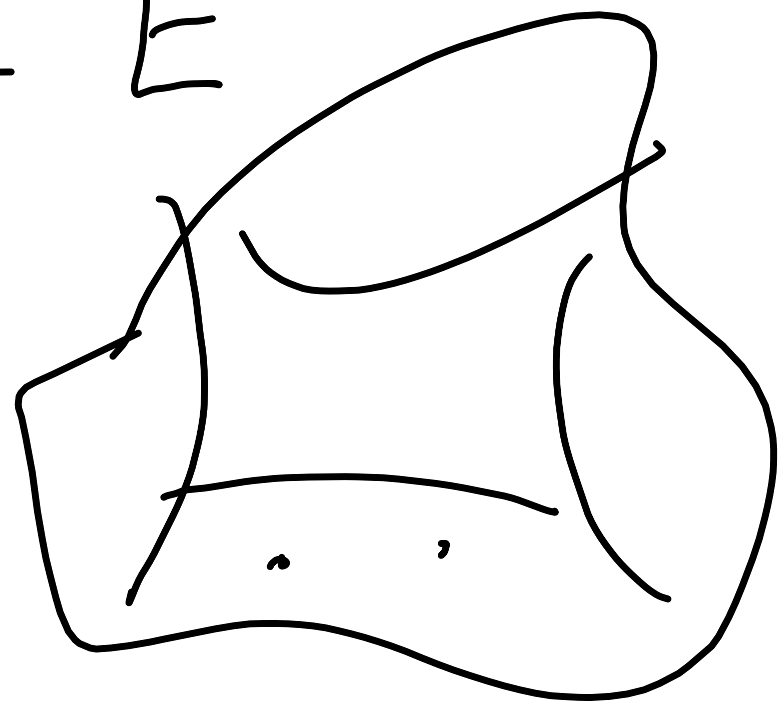
$\therefore$ $[x] = [y]$

If E is a partition of S

then $\exists \sim \subseteq S \times S$,

eq. rel. s.t.

$$E_{\sim} = E$$



Strings & Automata

Σ : a finite set.
alphabet.

Eg. $\Sigma = \{a, b, c\}$

String over Σ : a sequence of elements from Σ.

a,b,c   a, b, c, d

$f: [1, n] \to \Sigma$

a, b, c, d

abcd — Strings/Words.

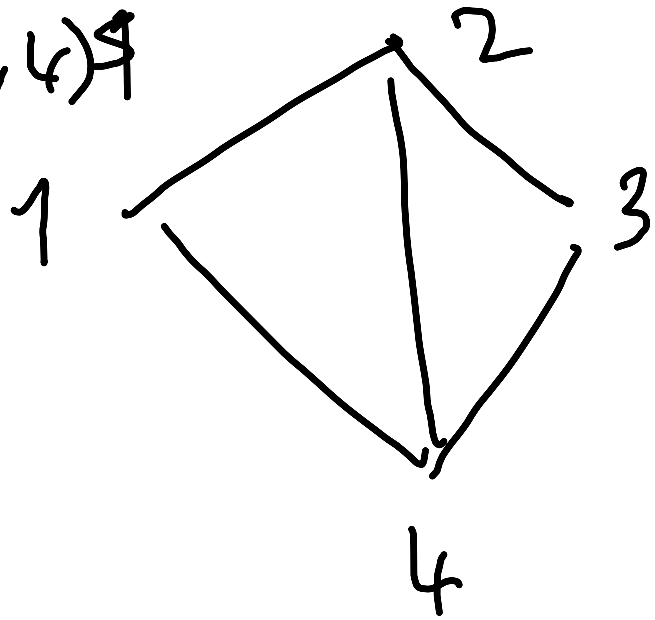
Input

Strings. $\to$ Computer (sorting algo) $\to$ Output.

0/1

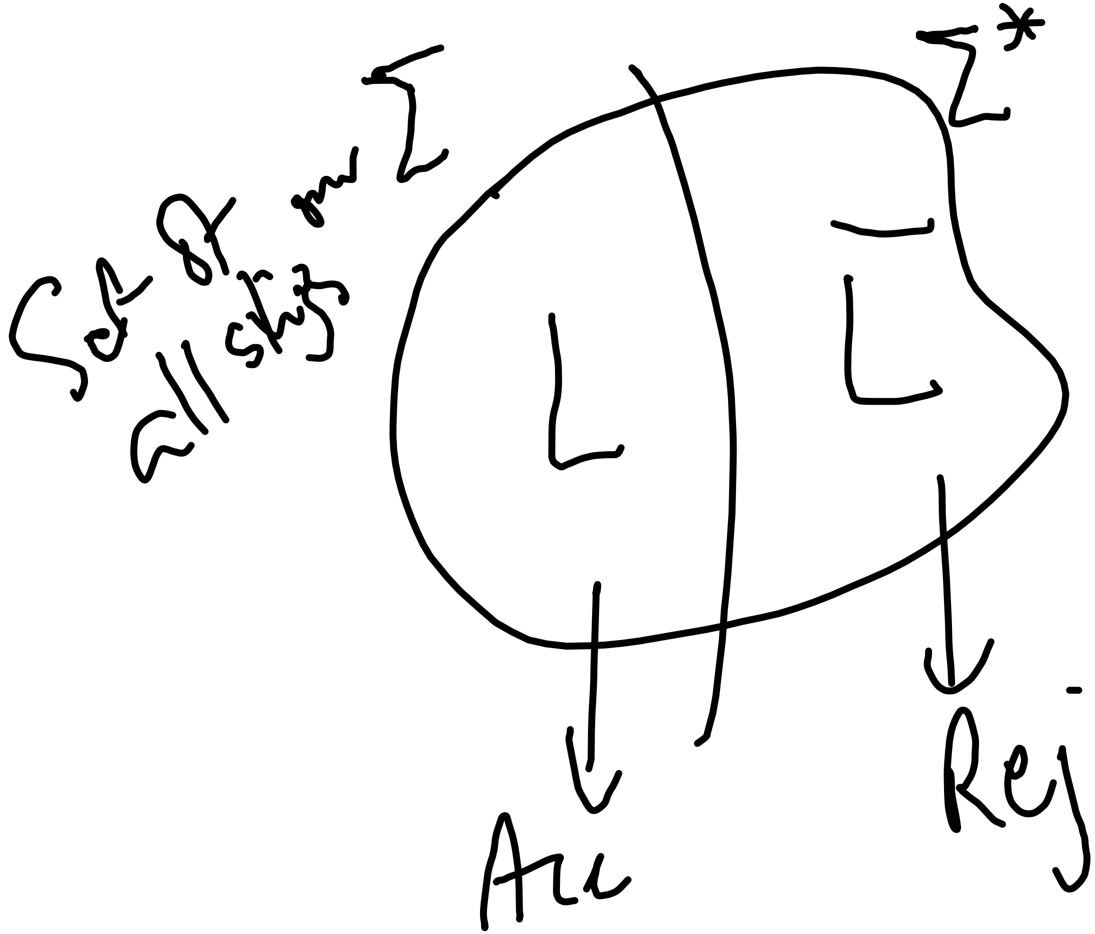
Graph

$1$2$3$4$$(1,2)(2,3)(3,4)(2,4)$

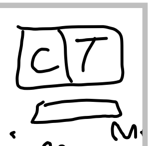


Strings over $\Sigma = \{0,1\}$

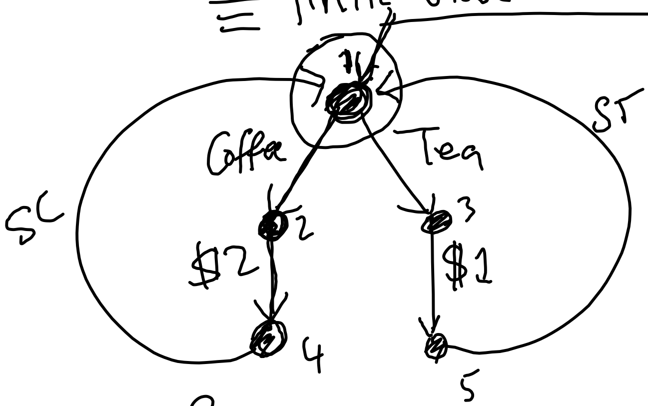
$L = \{ w \mid f(w) = Acc \}$


Set of all strings over Σ
Σ*
L
L̄
Acc
Rej

Automata



$\equiv$ finite-state machines



$\Sigma = \{$Coffee, Tea, \$1, \$2, SC, ST$\}$

Coffee . \$2 . SC — Acc ✓

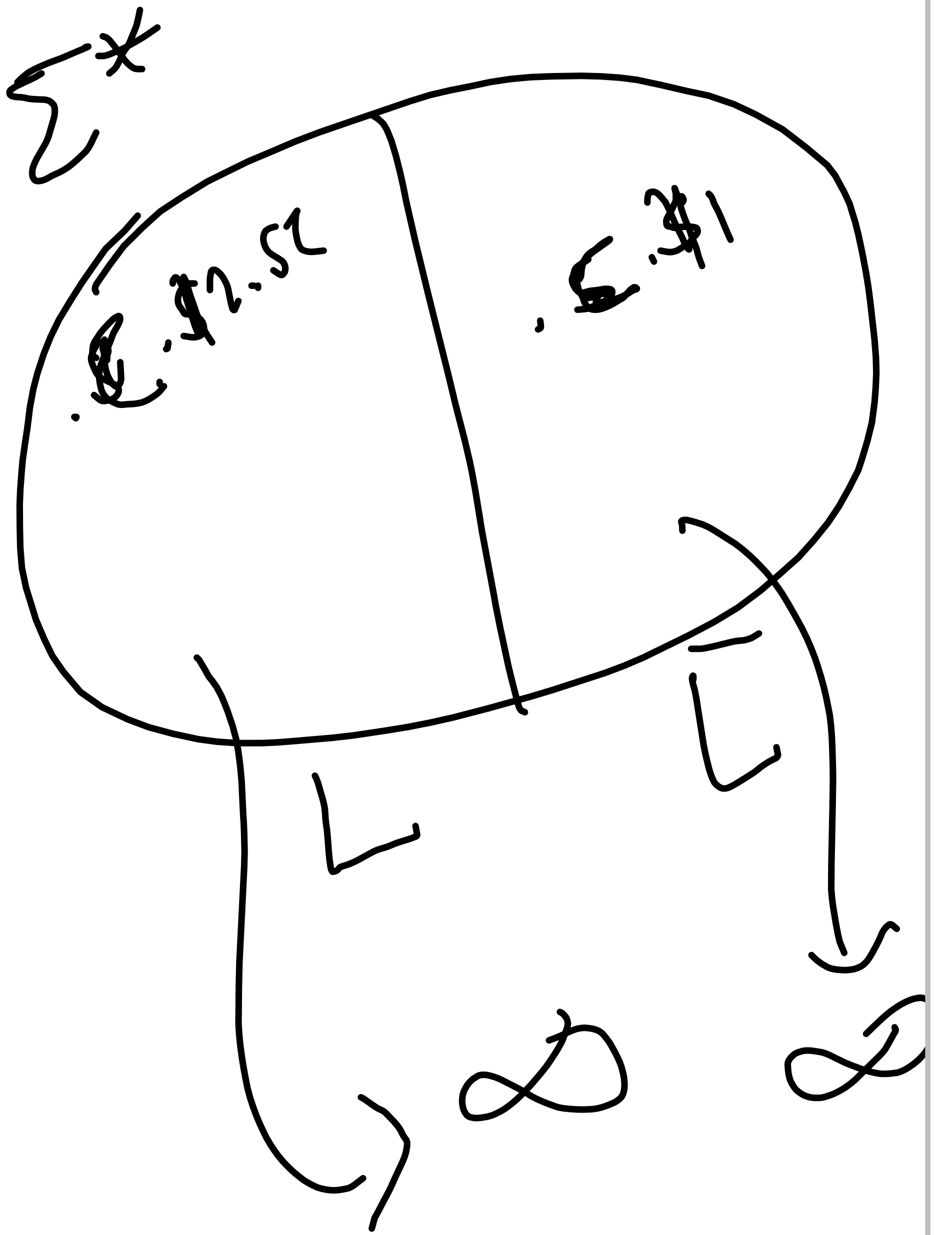
Coffee . \$1 — Rej

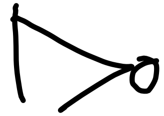
Coffee \$2 SC Tea \$1 ST . . .

$w \in \Sigma^*$ is accepted if there is a path in M on w

Coffee . \$2 — Rej

$w \in \Sigma^*$ is accepted if there is a path in M that reaches a marked/final state


Σ*



x		y
0	$\longrightarrow$	0
0	$\longrightarrow$	1
1	$\longrightarrow$	1
	$\longrightarrow$	0

$$\Sigma = \{ \langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle \}$$

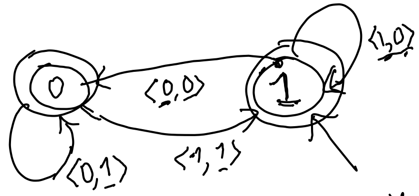
$$\langle x, y\rangle$$

$\neg$

$\langle 0,0\rangle \langle 0,1\rangle \langle 1,1\rangle \langle 1,0\rangle$ ______

$\langle 1,0\rangle \langle 1,0\rangle$ ______

M



$\langle 0,0\rangle \langle 0,1\rangle$

$$L = \{ \langle x_1, y_1\rangle \langle x_2, y_2\rangle \ldots \langle x_n, y_n\rangle \mid y_1 = 0,\ \forall i<n\ y_{i+1} = \overline{x_i} \} \cup \{\epsilon\}$$

$$\mathrm{Lang}(M) \subseteq L$$

Induction on $|w|$

Base case:
- $\varepsilon \in L(M)$, $\varepsilon \in L$
- $\langle x, y\rangle \in L(M)$: $\langle 0,0\rangle \in L$ or $\langle 1,0\rangle \in L$

Induction step $|w| > 1$

$\langle x_1, y_1\rangle \ldots \langle x_{n-1}, y_{n-1}\rangle \langle x_n, y_n\rangle$

Stronger Inductive hypothesis

$w \in L(M) \Rightarrow w \in L$

On $w = \langle x_1, y_1\rangle \ldots \langle x_n, y_n\rangle$ M reaches x_n

$\underline{\langle x_1, y_1\rangle \ldots \langle x_{n-1}, y_{n-1}\rangle} \langle x_n, y_n\rangle \in L(M)$

$M \xrightarrow{\langle x_1, y_1\rangle \ldots \langle x_{n-1}, y_{n-1}\rangle} x_{n-1} \longrightarrow y = \overline{x_{n-1}}$

$$y_n = \overline{x_{n-1}}$$

$\therefore \langle x_1, y_1\rangle \ldots \langle x_{n-1}, y_{n-1}\rangle \in L$

Also, on w, M reaches x_{n-1}.

$\Sigma = \{0, 1\}$

$L = \{ w \mid \#\ 0\text{'s in } w \text{ is even} \}$

$0 \notin L$

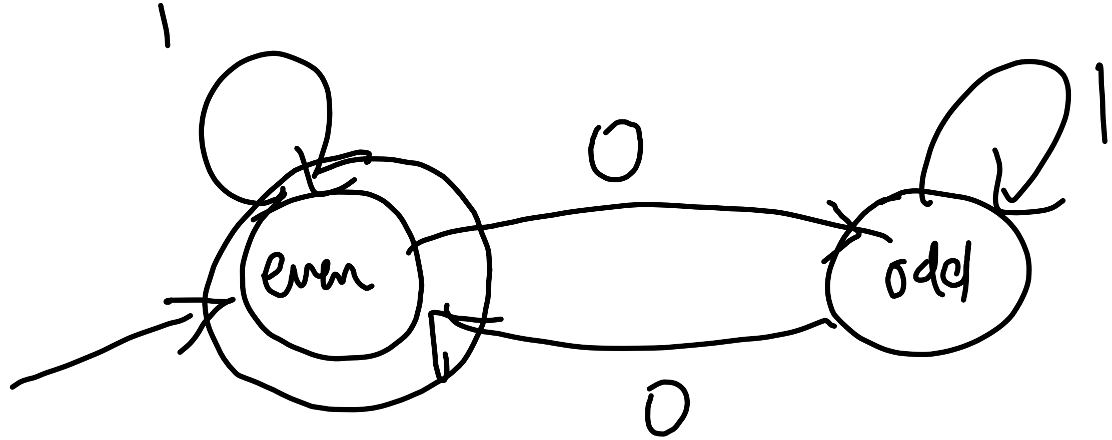
$01 \notin L$

$001 \in L$

$101 \notin L$

$1010 \in L$

M



$w \in L(M)$

marked

$\text{Init state} \xrightarrow{\quad w \quad} q$

$0 \notin L(M)$

$00 \in L(M)$

$L(M) = \{ w \mid \#0\text{'s in } w \text{ is even} \}$

Summary

Strings

Computations

f: Strings $\to \{0,1\}$

$$f \longleftrightarrow L \subseteq \Sigma^*$$

Notion of a state

Automaton – Finite state machine

Marked states are useful.