CS173 Discrete Mathematical Structures
Fall 2006
Homework #9
due Tuesday, November 7, 2006, 8:00 a.m.
(46 points + 6 bonus points)

1. The Pigeonhole Principle (10 points)

a. (5 points) There are 26 houses on a street. Each house has a distinct address between 1000 and 1049, inclusive. Show that at least two houses have addresses that are consecutive integers.

Hint: Group all available addresses into pairs of consecutive numbers. Take them as "pigeonholes".

b. (5 points) Prove that any set of 17 nonempty strings over {*a*, *b*, *c*, *d*} have two different strings whose starting letters agree and ending letters agree.

Hint: Nonempty strings over {*a*, *b*, *c*, *d*} can be any size greater than 0, can use any combination of letters, and can have any order. We are only concerned with the starting and ending letters of strings.

2. Combinatorial Proof (8 points)

Give a combinatorial proof that if n is a positive integer then $\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}$.

Hints:

1. Read *Chapter 5.4* in your textbook. There are a couple of nice combinatorial proofs. See *Corollary 1* and *Theorem 2*.
2. Consider a set A of n elements. Both sides count the ways to first select a subset S of A and then select two not necessarily distinct elements from S. By saying not necessarily distinct, it means that we select twice from S, and the selected element of each selection may be the same element. For example, we choose x for the first time, and again choose x for the second time.
3. Express the right-hand side as $n(n-1)2^{n-2} + n2^{n-1}$. Match these two terms with the following two situations: the two elements chosen are different, and the two elements chosen are the same.

3. Binomial Coefficients (10 points + 6 bonus points)

a. (5 points) Find the 2nd term of the expansion of $(x-\frac{1}{x^3})^{200}$ (note that we start counting from 0th term, then 1st term, 2nd term, 3rd term, etc.). Then find the coefficient of $\frac{1}{x^{300}}$ in this expansion. Is there a constant term in the expansion? If yes, find it. If no, give a brief explanation.

b. (Bonus: 6 points) Give a formula for the coefficient of x^k in the expansion of $(x-\frac{1}{x^3})^{200}$, where k is an integer.

Hints:

1. List a few terms of the expansion to find the property of k. What is the range of k? Are they a multiple of some integer?
2. Your formula should look like a piecewise function.
3. Pay attention to the +/- sign of each term.

c. (5 points) Use the binomial theorem to prove that

$$C(n,0)-C(n,1)+C(n,2)-\cdots+(-1)^n C(n,n)=0$$

4. Combination & Permutation (18 points)

For each of the following question, show the expression. Do not calculate the final solution.

a. (10 points) How many bit strings (strings over {0, 1}) of length 12 contain

1. exactly three 1s?
2. at most three 1s?
3. at least four 1s?
4. an equal number of 0s and 1s?
5. more 0s than 1s?

b. (8 points) How many strings of six lowercase letters from the English alphabet contain

1. the letter x?
2. the letters x and y?
3. the letters x and y in consecutive positions with x preceding y, with all the letters distinct?
4. the letters x and y, where x is somewhere to the left of y in the string, with all the letters distinct?