

CS 173: Midterm Exam 1

Fall 2006

Name: _____

NetID: _____

Section Leader: _____

General Directions

1. Make sure your name is on every page.
2. Remember to write clearly and legibly. Unreadable answers will receive no credit.
3. This is a closed book exam. No notes of any kind are allowed. No calculators.
4. Remember to time yourself.

Question	Points	Out of
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		15
10		15
11		15
12		15
Total		100

Multiple Choice

Problem 1 (5pts)

Which of these disjunctions is false?

- a) $(1 + 1 = 2) \vee (2 + 2 = 5)$
- b) $(1 + 1 = 3) \vee (2 + 2 = 5)$
- c) $(1 + 1 = 3) \vee (2 + 2 = 4)$
- d) $(1 + 1 = 2) \vee (2 + 2 = 4)$

Solution: B

Problem 2 (5pts)

Which of these propositions is not logically equivalent to the other three?

- a) $(p \vee r) \rightarrow q$
- b) $(p \rightarrow q) \wedge (r \rightarrow q)$
- c) $(p \wedge r) \rightarrow q$
- d) $\neg q \rightarrow (\neg p \wedge \neg r)$

Solution: C

Problem 3 (5pts)

Which of these propositions has a truth value different from the other three (the domain is the set of real numbers)?

- a) $\forall x \exists y (x \neq 0 \rightarrow x \cdot y = 1)$
- b) $\exists y \forall x (x + y = x)$
- c) $\forall x \forall y [(x \neq y) \rightarrow \exists z (x < z < y \vee y < z < x)]$
- d) $\forall x \forall y \exists z (x < z < y)$

Solution: D

Problem 4 (5pts)

Suppose $A \subseteq B \subseteq C$. If $S = C - A$ and $T = C - B$, then which of the following best describes the relationship between S and T ?

- a) $S \subseteq T$
- b) $T \subseteq S$
- c) $S = T$
- d) It cannot be determined from the information given.

Solution: B

Problem 5 (5pts)

Which of the following does not necessarily imply that $A \cup B = A$? (The symbol 2^A denotes the power set of A .)

- a) $B \in 2^A$
- b) $B \subseteq 2^A$
- c) $B = \emptyset$
- d) $B \subseteq A$

Solution: B

Problem 6 (5pts)

If $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions, which of the following is not necessarily true?

- a) If $g \circ f$ is one-to-one, then f is one-to-one.
- b) If $g \circ f$ is onto, then g is onto.
- c) If $g \circ f$ is one-to-one, then g is one-to-one.
- d) $g \circ f \subseteq A \times C$

Solution: C

Problem 7 (5pts)

Which one of these propositions is different from the other three? (For this problem, f and g are non-negative functions.)

- a) $g(x) = \Omega(f(x))$
- b) $\exists c \exists k [(x > k) \rightarrow (f(x) \leq c \cdot g(x))]$
- c) $\forall c \exists k [(x > k) \rightarrow (f(x) \leq c \cdot g(x))]$
- d) $\neg \forall c \forall k [(x > k) \wedge (f(x) > c \cdot g(x))]$

Solution: C or D

Problem 8 (5pts)

Let $f(x) = 3x - 5$ and $g(x) = \frac{x+5}{3}$ be functions defined on the real numbers ($f : \mathbb{R} \rightarrow \mathbb{R}, g : \mathbb{R} \rightarrow \mathbb{R}$). Which of the following is not true?

- a) $f(x) = O(g(x))$
- b) $f(x) = g^{-1}(x)$
- c) $g \circ f(x) = x$
- d) $\neg \exists x (f(x) = g(x))$

Solution: D

Proofs**Problem 9 (15pts)**

Prove that $3n^2 + n \log n$ is $O(n^2 - n)$.

Solution:

I prove that for $c=5$ and $n \geq 5$

$$3n^2 + n \log n \leq c(n^2 - n)$$

Proof:

$$3n^2 + n \log n \leq 5(n^2 - n)$$

$$0 \leq 5n^2 - 5n - 3n^2 - n \log n = 2n^2 - 5n - n \log n$$

Since $n \geq 5$, $\log n \leq n$ and $5n \leq n^2$. Thus:

$$2n^2 - 5n - n \log n \geq 2n^2 - n^2 - n^2 \geq 0$$

Grading rubric:

- * Not using or not understanding the definition of big-Oh: -10.
- * Using limits (unless personally approved by your TA): -10.
- * Proving a simpler problem (e.g: $3n^2 + n \log n$ is $O(n^2)$): from -5 to -8.
- * Claiming you can drop 'insignificant terms' (like $-n$): -10.
- * Not proving inequalities (e.g: claiming that $3n^2 \leq 4(n^2 - n)$ without proof): from -5 to -2.
- * Plugging in $c=10, k=100$ or fixing $k=100$ and finding c , but not proving for $n \geq k$ part: -7.
- * Math errors: from -1 to -2.
- * Participation: 2 points.
- * Sense of humor: +1.

Problem 10 (15pts)

Prove that if $f : A \rightarrow B$ is a one to one function, and S and T are subsets of A , then $f(S \cap T) = f(S) \cap f(T)$.

Solution:

Assume that $f : A \rightarrow B$ is a one to one function, and S and T are subsets of A , and prove that $f(S \cap T) = f(S) \cap f(T)$ (direct proof).

Notice that the left side and the right side of the equal sign are simply subsets of the co-domain. We know that one way of proving that two sets are equal is to show that they are subsets of one another.

First, show that $f(S \cap T) \subseteq f(S) \cap f(T)$. Let y be an arbitrary element of $f(S \cap T)$. We'll show that it must also be an element of $f(S) \cap f(T)$ (standard technique for proving subset... it comes straight from the definition!). If $y \in f(S \cap T)$ then there $\exists x \in S \cap T$ such that $f(x) = y$ (definition of function). Thus, $x \in S$ and $x \in T$ (definition of intersection). $x \in S$ implies $y \in f(S)$ and $x \in T$ implies $y \in f(T)$. Thus, $y \in f(S) \cap f(T)$.

Second show that $f(S) \cap f(T) \subseteq f(S \cap T)$. Let y be an arbitrary element of $f(S) \cap f(T)$. Then $y \in f(S)$ and $y \in f(T)$. If $y \in f(S)$ then $\exists x_1 \in S$ such that $f(x_1) = y$. Similarly, $y \in f(T)$ implies $\exists x_2 \in T$ such that $f(x_2) = y$. Since f is one to one, $x_1 = x_2$, and $x_1 \in S$ and $x_1 \in T$. By definition of intersection, $x_1 \in S \cap T$, and thus, $y \in f(S \cap T)$ (since $y = f(x_1)$).

Rubric: I awarded 7 points for each of the two directions, plus a point for any meaningful setup. Failure to mention that f was one to one in the second portion was a deduction of 4 points. Otherwise, each step in each of the subproofs was approximately 2 points.

In general, scores for this problem were very low.

Problem 11 (15pts)

- a) Prove that if n is an integer and $n^3 + 5$ is odd, then n is even, using a proof by contradiction or an indirect proof.

Solution: Proof by contradiction. Assume n is odd. According to the definition of an odd number, $n = 2k + 1$ where k is a nonnegative integer. Therefore, we have

$$\begin{aligned} n^3 + 5 &= (2k + 1)^3 + 5 \\ &= (8k^3 + 12k^2 + 6k + 1) + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2(4k^3 + 6k^2 + 3k + 3) \end{aligned}$$

Let $j = 4k^3 + 6k^2 + 3k + 3$, then $n^3 + 5 = 2j$ and therefore $n^3 + 5$ is an even number. This is in contradiction with the fact that $n^3 + 5$ is an odd number. Thus, we have proved n is even.

Rubric:

8 points in total.

2 points: structure of an indirect proof or proof by contradiction. Specifically, assume n is odd (or assume the contrapositive, if n is odd, then $n^3 + 5$ is even).

2 points: definition of even/odd numbers. An odd number: $n = 2k + 1$ where k is a nonnegative integer. An even number: $n = 2k$ where k is a nonnegative integer.

3 points: expansion of $n^3 + 5 = (2k + 1)^3 + 5$.

1 point: miscs.

Note: an example, say, assume $n = 1$, is not a proof. It is not sufficient to prove the general case. 6 or 7 points will be deducted in this case.

- b) Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. Use a proof by cases, where the two cases are " $x \geq y$ " and " $x < y$ ".

Solution: Proof by cases.

Case 1: $x \geq y$. Then we have $\max(x, y) = x$ and $\min(x, y) = y$.
 $\max(x, y) + \min(x, y) = x + y$.

Case 2: $x < y$. Then we have $\max(x, y) = y$ and $\min(x, y) = x$.
 $\max(x, y) + \min(x, y) = y + x = x + y$.

Rubric:

7 points in total.

1 point for structure of the proof. Specifically, you need two parts, $x \geq y$ and $x < y$.

3 points for each part.

Note: an example, say, let $x = 3$ and $y = 2$ is simply not a proof. 6 points will be deducted if only examples are shown in the proof.

Problem 12 (15pts)

- a) Let j and k be integers, with j even and k odd. Prove that the product of j and k is even.
- b) Prove that the product of consecutive integers is even. Hint: you can use part a) in your solution.
- c) Prove that the square of an odd integer equals $8k + 1$ for some integer k . Hint: you can use part b) in your solution.

Solution:

- a) Let $j = 2x$ and let $k = 2y + 1$ for integers x, y . Then the product of j and k is:

$$jk = (2x)(2y + 1) = 2(xy + x)$$

Since the product and addition of integers is an integer, $xy + x$ is an integer. Therefore, we have expressed jk as twice an integer, and we conclude that jk is even by definition.

Rubric: 5 points total. Potential point losses: 3 points for using the same variable in representing j and k (it was a VERY common mistake to use the same variable, which implies that j and k are consecutive), 3 points for using modulus (it throws away too much information), up to all 5 points for an unclear or invalid argument.

- b) Looking at any pair of consecutive numbers, either the first is even and the next is odd, or the first is odd and the next is even. Either way, we are taking the product of an odd integer and an even integer, and thus we can apply part a. Their product must be even.

Rubric: 3 points total. Potential point losses: 2 for not remembering to state both cases (it was a relatively common mistake to assume that the first number of the consecutive pair was even, as the first listed integer in the first part of the problem was even), 1 point for failing to come to a conclusion in referencing part a when necessary.

- c) Let our odd integer be $a = 2b + 1$. Then the square of our integer is:

$$a^2 = (2b + 1)^2 = 4b^2 + 4b + 1 = 4b(b + 1) + 1$$

As you can see, we have factored out a $4b$ from part of the sum. But now observe that we are taking the product of b and $b + 1$, two consecutive integers. By part b, we know that this product is even, and by the definition of even, it can be represented as twice an integer, say k . Therefore, as desired:

$$a^2 = 4(2k) + 1 = 8k + 1$$

Rubric: 5 points total. Potential point losses: 3 for not giving a proper and logical conclusion of proof (for example, claiming that some number divided by another results in an integer, k , without proof), 3 points for failing to prove the statement for ANY odd integer (the most common error was letting the odd number be $2k + 1$, squaring it, setting it equal to $8k + 1$, and solving for k), 1 point for not referencing the proof in part b where necessary.

Also, 2 points total for properly declaring when variables need to be integers, such as x and y in the solution to part a or b in the solution to part c.

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