CS 598 ANH - Calculus on Meshes

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Homework 2

Due : Tuesday, October 4, 2005 by end of class time.

- **Problem 1:** (Vector Fields) Consider the vector field X on $\mathbb{R}^2$ defined as $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Compute the flow of this vector field for time $t \in \mathbb{R}$.

- **Problem 2:** (Duality) (a) Let V be an n-dimensional vector space, $v_1, \ldots, v_n$ a basis for this vector space and $\alpha_1 \ldots \alpha_n$ a set of n real scalars. Show that there is one and only one real valued linear function $f : V \to \mathbb{R}$ such that $f(v_i) = \alpha_i$. (b) Let $f_1 \ldots f_n \in V^*$ such that $f_i(v_j) = \delta_{ij}$ (recall that V^* is just the space of real valued linear functions on V). Show that the f_i form a basis for V^*.

- **Problem 3:** (Tensors) Let V be a vector space and $\phi \in T^0_k(V)$ an $(0, k)$-tensor (i.e it takes k vector arguments). Define the alternation ($\mathbf{A}$) and symmetrizing ($\mathbf{S}$) maps as

$$(\mathbf{A}\phi)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$
$$(\mathbf{S}\phi)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\,.$$

  Show that these define an alternating and a symmetric tensor, respectively.

- **Problem 4:** (Exterior Derivative) (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the map $f(x, y) = x^2 + y^2$. Compute the exterior derivative $\mathbf{d}f$. Compute $\mathbf{d}(\mathbf{d}f)$. (b) Now consider the clockwise oriented equilateral triangle on the plane, with the 3 vertices v_0, v_1, v_2 being $(0, 0)$, $(0.5, \sqrt{3}/2)$, $(1, 0)$ respectively. Discretize f by sampling it at the vertices. Orient the edges as $[v_0, v_1]$, $[v_1, v_2]$, $[v_2, v_0]$. Compute $\mathbf{d}f$ and $\mathbf{d}(\mathbf{d}f)$ using the definition of discrete exterior derivative. (c) Discretize $\mathbf{d}f$ by integrating on the oriented edges, the smooth $\mathbf{d}f$ you obtained in (a). (d) Let $\varphi_i, i = 0, 1, 2$ be the real valued affine functions defined on the plane which take value 1 on vertex i and 0 on the other two vertices. Let $\hat{f} = \sum_{i=0}^{2} \varphi_i f_i$ be the linearly interpolated function obtained from the vertex sampled values of f. Compute $\nabla \hat{f}$ in the triangle. Integrate the vector field $\nabla \hat{f}$ along each of the oriented edges of the triangle.

- **Problem 5 :** (Hodge Star) Show that equation (6.4.1) in my thesis follows from equation (6.4.2).
