CS 598 ANH - Calculus on Meshes

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Homework 1

Due : Tuesday, September 20, 2005 by end of class time.

- **Problem 1:** (Analysis) Consider $\mathbb{R}$ as a topological space with the topology induced by the usual distance metric (so the open sets of the topology are open intervals and unions of open intervals). Consider the functions $f : \mathbb{R} \to \mathbb{R}$. Prove that the two definitions of continuity given in class are equivalent. That is, a function f is continuous in the sense of metric spaces if and only if it is continuous in the sense of topological spaces.

- **Problem 2 :** (Smooth Manifolds) Consider S^2, a 2-sphere (two dimensional sphere) of radius 1 embedded in $\mathbb{R}^3$ with its center at the origin. Define two open sets U_n and U_s such that U_n is all of the sphere except the north pole and U_s is all of the sphere except the south pole (the north pole is at $(0, 0, +1)$ and the south pole is at $(0, 0, -1)$). Define two maps $\varphi_n : S^2 \to \mathbb{R}^2$ and $\varphi_s : S^2 \to \mathbb{R}^2$ where $\mathbb{R}^2$ is the xy-plane . The map φ_n is defined as follows : for a point $p \in S^2$ other than the north pole, $\varphi_n(p) \in \mathbb{R}^2$ is the point in the xy-plane where the ray from the north pole to p intersects the xy-plane. The map φ_s is defined analogously. Show that this defines a system of C^∞ charts (this is known as the stereographic projection).

- **Problem 3 :** (Simplices) (a) Show that the definition of a simplex given in class defines a triangle if 3 vertices are used in the definition. (b) Recall that a triangle is well-centered (its circumcenter lies inside it) if and only if all its angles are less than $\pi/2$. Prove this. (c) Can you find a simple if and only if condition like this (involving angles, lengths or areas) for deciding if a tetrahedron is well-centered ?

- **Problem 4:** (Oriented Dual Meshes) Consider a tetrahedral mesh embedded in $\mathbb{R}^3$ such that all the tetrahedra are oriented by the right hand rule. Let $[v_0, v_1, v_2, v_3]$ be an oriented tetrahedron in this mesh. Determine the orientation of the *dual* of the edge $[v_0, v_1]$ (recall that the dual will be a two dimensional surface). Draw a clear figure with the simplicial subdivision to show your work. Show orientations using arrows. Also give the algebraic derivation of your answer.

- **Problem 5 :** (Vector Fields) Let $X, Y, Z \in \mathfrak{X}(M)$ be smooth vector fields on a manifold M. Prove that they satisfy the Jacobi identity.
