University of Illinois at Urbana-Champaign
Department of Computer Science

Midterm 2

CS 273 Introduction to Theoretical Computer Science
Fall 2005

Name:
Netid:

- Print your name and netid, *neatly* in the space provided above; print your name at the upper right corner of *every* page. Please print legibly.
- This is a *closed book* exam. No notes, books, dictionaries, or calculators are permitted.
- Write your answers in the space provided for the corresponding problem. Let us know if you need more paper.
- Suggestions: Read through the entire exam first before starting work. Do not spend too much time on any single problem. If you get stuck, move on to something else and come back later.
- If you run short on time, remember that partial credit will be given.
- If any question is unclear, ask one of us for clarification.
- Unless otherwise stated, unsimplified expressions (in terms of factorials, sums, etc.) are acceptable.

Question	Points	Score	Grader
Problem 1	15		
Problem 2	20		
Problem 3	15		
Problem 4	15		
Problem 5	15		
Problem 6	20		
Total	100		

1. Short Problems (15 points)

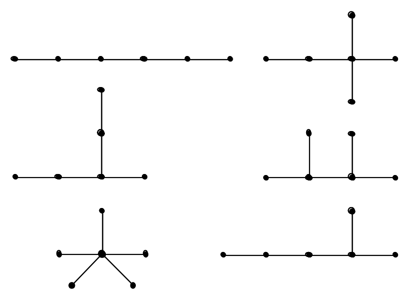
   Give brief answers to each of the following questions.

   (a) How many distinct trees can be made from 1000 labeled vertices? (3 points)

   Solution: By Cayley's theorem / Prüfer codes, as $n = 1000$ then $n^{n-2} = 1000^{998}$.

   (b) Show all non-isomorphic trees that can be made from 6 vertices. (3 points)

   Solution:



   (c) If a polyhedron has 20 vertices and 30 edges, how many faces must it have? (3 points)

   Solution: By Euler's formula $n - e + f = 2$, with $e = 30$ and $n = 20$ we have $f = 2 + 30 - 20 = 12$.

   (d) Give a tight asymptotic bound for the recurrence $T(n) = 5\,T(n/5) + 25n$. (3 points)

   Solution: By Masters theorem, $a = 5$, $b = 5$, $f(n) = 25n$ then as $n^{\log_5 5} = \Theta(n)$ the third case holds and $T(n) = \Theta(n \log n)$.

   (e) Give a tight asymptotic bound for the recurrence $T(n) = \frac{3}{2}\,T(n/8) + n^{1/3}$. (3 points)

   Solution: By Masters theorem, $a = \frac{3}{2}$, $b = 8$, $f(n) = n^{1/3}$ then as $n^{\log_8 \frac{3}{2} + \epsilon} < \Theta(n^{1/3})$ for all $0 < \epsilon < \frac{1}{2}$ we have $f(n)$ dominating the recurrence, thus $T(n) = \Theta(n^{1/3})$.

2. Linear Recurrences (20 points)

   (a) (10 points) Solve the following recurrence exactly:

   $$a_i = 6a_{i-1} + 7a_{i-2}.$$

   The initial conditions are $a_0 = 1$ and $a_1 = 2$.

   Solution: By the annihilator method,

   $$\begin{aligned} a_i - 6a_{i-1} - 7a_{i-2} &= 0 \\ E^2 - 6E - 7 &= 0 \\ (E-7)(E+1) &= 0 \end{aligned}$$

   Thus the generic form solution is $a_i = \alpha 7^i + \beta(-1)^i$. Using the intial conditions, we have the following system of equations:

   $$\begin{aligned} 1 &= \alpha + \beta \\ 2 &= 7\alpha - \beta \end{aligned}$$

   By substitution, $\beta = 1 - \alpha$ and $2 = 8\alpha - 1$. Thus $\alpha = \frac{3}{8}$ and $\beta = \frac{5}{8}$. Therefore the resulting closed form solution becomes

   $$a_i = \frac{3}{8} \cdot 7^i + \frac{5}{8} \cdot (-1)^i$$

   (b) (10 points) Give the general form of the solution to the following recurrence:

   $$a_i + 3a_{i-1} + 3a_{i-2} + a_{i-3} = 2^{i+1} - 3^i$$

   Express your answer using undetermined constants.

   Solution: By the annihilator method, first dealing with the homogeneous recurrence we have

   $$\begin{aligned} a_i + 3a_{i-1} + 3a_{i-2} + a_{i-3} &= 0 \\ E^3 + 3E^2 + 3E + 1 &= 0 \end{aligned}$$

   Notice that $(E+1)(E^2+2E+1) = E^3 + 3E^2 + 3E + 1$. As this is the case, and $E^2 + 2E + 1 = (E+1)^2$, the resulting annihilator is $(E+1)^3$.

   For the nonhomogeneous portion, notice that $(E-2)\langle 2^{i+1}\rangle = \langle 0\rangle$ and $(E-3)\langle 3^i\rangle = \langle 0\rangle 0$. Thus, the combined annihilator for the recurrence above is $(E+1)^3(E-2)(E-3)$ yielding the generic form solution of

   $$a_i = (\alpha i^2 + \beta i + \gamma)(-1)^i + \delta 2^i + \epsilon 3^i$$

3. Induction on Graphs (15 points)

   Prove by induction that every closed walk of odd length contains a cycle of odd length.

   If you are hoping for partial credit, be very explicit about the logic of the proof, what needs to be assumed, what needs to be shown, etc. No partial credit will be given for proofs that do not use induction.

Solution:

Base Case: A smallest odd length walk is of size 1 which loops on a given vertex. Clearly, this loop is an odd length cycle.

Inductive Hypothesis: For all closed walks of odd length less than $2k+1$, the walk contains an odd length cycle.

Inductive Step: Consider an odd length closed walk w of size $2k+1$. Clearly if the walk contains no repeated vertices (except the duplicate start and finish) then the walk is a cycle itself with odd length. If the walk constains some vertex v which is visited more than once (or the starting and ending vertex appears more than twice), namely $w = u \rightsquigarrow v \rightsquigarrow v \rightsquigarrow u$, consider the two resulting walks $v \rightsquigarrow v$ and $u \rightsquigarrow v \rightsquigarrow u$ by removing the internal closed walk containing v. As the length of the original walk was odd, then one of the two resulting walks is also odd and by the inductive hypothesis contains an odd cycle. As adding extra edges to the odd walk does not remove the cycle, then w also contains an odd cycle.

Therefore, by induction any odd length closed walk contains an odd cycle.

4. Planar Graph Theory (15 points)

Prove that every polyhedron has at least one face that has 5 or less edges forming its boundary.

Solution: Consider the planar graph G of the polyhedron. Let F_i = the number of faces of G of bound degree i, and V_i = the number of vertices of degree i. Since every edge of G borders exactly two faces, we get

$$2E = \sum_{i \geq 3} iF_i$$

If every face had 6 or more edges, then

$$2E = \sum_{i \geq 3} iF_i \geq \sum_{i \geq 6} 6F_i = 6F$$

implying $F \leq \frac{1}{3}E$. Furthermore, because each vertex has degree at least 3,

$$2E = \sum_{i \geq 3} iV_i \geq 3V$$

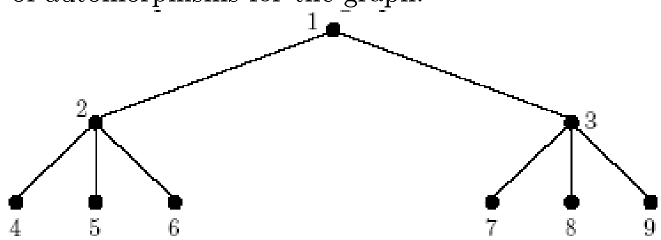
implying $V \leq \frac{2}{3}E$. Therefore

$$E = V + F - 2 \leq \frac{2}{3}E + \frac{1}{3}E - 2 = E - 2$$

A contradiction. Hence at least one face has 5 or less edges forming its boundary.

5. Automorphisms (15 points)

Determine the number of automorphisms for the graph:



and clearly explain why this number is correct.

Solution: As vertex 1 is the only vertex of degree 2, it can never be permuted. As vertices 2 and 3 both have degree 4 and have similar subtrees, each can be swapped. Similarly, the sets of leaves which share a parents, namely 4,5,6 and 7,8,9 can each be swapped in any way with siblings. Thus, the number of automortphisms of the above graph is $2 \cdot 3! \cdot 3! = 72$.

6. Divide-and-Conquer Recurrences (20 points)

Determine a tight asymptotic bound for the recurrence,

$$T(n) = n^2\, T(n/2) + 1,$$

and show your work.

Solution: Consider the recurrence tree for the above. Notice that at level $i-1$ a node has $\left(\frac{n}{2^i}\right)^2 = \frac{n^2}{2^{i+1}}$ children. As each node does constant work, the work for the entire tree consists of

$$\sum_{i=0}^{d} \prod_{j=0}^{i} \frac{n^2}{2^{j+1}}$$

where d is the depth of the tree. As each recursive call is half the size of the previous, $d = \log n$ and

$$\sum_{i=0}^{\log n} \prod_{j=0}^{i} \frac{n^2}{2^{j+1}}$$

Bounding the sum, we can consider only the final row (largest element in the polynomial), namely $\frac{n^{2\log n}}{2^{\log n+1}} \geq \frac{n^{2\log n}}{2n} = \frac{n^{2\log n - 1}}{2}$. Thus $T(n) = \Omega(n^{2\log n}$.

Similarly, a weak upper bound can be given by merely dropping the divisor from the sum. The resulting polynomial is then dominated by the last term, namely $n^{2\log n}$. Thus $T(n) = O(n^{2\log n}$.

As the upper and lower bounds agree, $T(n) = \Theta(n^{2\log n})$.